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BOUNDS AND ROBUST HEDGING OF THE AMERICAN OPTION

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The value of an American option depends on the information that the holder will acquire over the option's life. Much of the literature makes restrictive assumptions about information revelation – for example that the underlying price process is Markov. With a richer information structure, the American feature becomes more valuable. This paper identifies the least upper bound on the price of an American option, placing no assumptions on the information structure. It shows that the American premium in standard models is a small fraction of its upper bound, and shows what features make the American feature most valuable. The bounds can be tightened by excluding implausible processes, and these bounds are enforced by a hedging strategy that is robust to model error.

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1. INTRODUCTION

American options are valuable precisely because the holder is free to take the exercise decision using all information about future returns that arrives over the option's life. The more information the holder receives, the better the exercise decision, and the more valuable the timing option. Yet much of the extensive literature on American options makes the extreme assumption that the underlying asset price follows a Markov process, and that all information is contained in the current price. With a richer information structure, where the holder learns about the distribution of future returns, the value of being American will in general be higher. We investigate how great this extra value could be by exploring the arbitrage-free upper bound on the price of an American option given only the contemporaneous prices of all European options. The bound is enforced by a semi-static hedging strategy. We identify the hedging strategy, and show how it can be improved, reducing its cost while retaining its robust properties.

American options are the most common and also the most subtle of exotic options. While many exotic options have pay-offs that are functions of the price path alone, the pay-off to an American option depends also on the exercise strategy of the holder. The decision to exercise will depend on the holder's beliefs about the distribution of future returns at each point over the life of the option. In the standard Brownian diffusion model, the only relevant information for the holder of the option is the current asset price. The optimal exercise strategy is to exercise the option the first time the asset price breaches a barrier – the *immediate exercise boundary* (see Broadie and Detemple (2004) for a more formal treatment). With a richer information structure – for example with stochastic volatility – the exercise strategy is more complex, and depends on all state variables rather than just the underlying asset price.

Adding further state variables gives only limited insight into the value of flexibility. Formal models can only take account of what might be called known unknowns (Rumsfeld 2002); the value of being American is that the holder is able to respond to any new information that becomes available prior to expiry – including information on the unknown unknowns.

In general, holding the price of European options fixed, the richer the information structure, the greater the value of the American option. As Longstaff, Santa-Clara and Schwartz (2001) argue in the context of American and Bermudan style swaptions, the costs of using a single-factor model to make exercise decisions when the term structure is driven by multiple factors are substantial. They claim that these costs remain substantial even if the model is recalibrated to fit the current set of market prices of related derivative securities (in this case European swaptions) at each possible exercise date. While Andersen and Andreasen (2001) and Svenstrup (2005) contest this latter conclusion, it is common ground that a single factor model has to be recalibrated to take account of new information revealed in the market prices of other derivative securities on the same underlying when deciding whether to exercise the option.

Repeated recalibration of a model is internally inconsistent. It violates the model's own assumption that the parameters are constant. Buraschi and Corielli (2005) show that it is precisely in the non-Markovian setting in which we are interested that the improvements from recalibration are smallest.

We therefore follow a different route. To get a bound on how much new information could matter, we ask: given the prices of European options, how much could an American option be worth without allowing arbitrage opportunities? In answering the question, the only assumption we make (apart from the standard assumptions of frictionless markets and absence of arbitrage) is that interest rates and dividends are deterministic.

The search for arbitrage-based option pricing bounds goes as far back as Merton (1973) who shows that the price of an American put option with strike K and maturity T is bounded below by the equivalent European put option; using obvious notation, $P^a(K, T) \geq P^e(K, T)$. Brown, Hobson, and Rogers (2001) generalize the question of pricing bounds by searching for the tightest bounds on a claim (in their case barrier and lookback options) in the presence of a whole set of other claims (European options with the same maturity). These bounds are useful because they are enforced by hedging strategies that are robust, strategies that work however the prices of the instruments used

for hedging evolve. Robust strategies put a floor on potential hedging losses. Being model free, they avoid the problems highlighted by Green and Figlewski (1999) when conventional hedging techniques are used with an incorrect model or with poor parameter estimates.

The bound on the value of the American feature is not in general very close to the price from standard pricing models. As will be shown in the case of the American put option, the maximum possible value of the American premium (defined as the premium compared with the most valuable European option with the same strike and the same or shorter maturity) can be three times greater than that implied by standard diffusion models without creating an arbitrage opportunity. Models with richer information structures may generate American option values that differ substantially from traditional models, and hedging American options using standard models may lead to significant error.

The rational bound is enforced by a hedging strategy that is robust, in the sense that the maximum loss the seller of the option can incur is strictly limited however the asset price moves. This desirable feature is offset by the fact that the maximum loss can be large, and alternative hedging strategies may have much smaller mean square error. We show how the strategy can be improved. By placing weak restrictions on prices at which options will trade in the future, we show how the maximum loss and the mean square hedging can be significantly reduced, albeit with some slight loss of robustness.

We use linear programming to find the rational bounds, having first set the problem in a discrete time, discrete price framework. Exploiting the duality between pricing and hedging, we show that the cheapest portfolio that dominates the American option is also the price of the option under the process that values it most highly. Others authors (Andersen and Broadie (2004), Rogers (2002) and Haugh and Kogan (2004)) exploit the relationship between the primal problem of pricing, and its dual problem of hedging to bound the value of an American option. They seek pricing bounds that bracket the true value of the American option under a known price process by using a near to optimal

exercise strategy. We differ fundamentally from them in seeking price bounds that are independent of the price process.

The paper is organized as follows. The upper bound on the general American option is identified in section 2. Section 3 computes the bound on the price of the American put option and explores the type of processes that make the American feature most valuable. Section 4 tightens the bounds by imposing some restrictions on the future level of implied volatility. In section 5, simulation is used to model the rational bounds hedges and compare them with conventional delta-gamma hedges. The final section concludes.

2. IDENTIFICATION OF THE BOUNDS

2.1 An informal overview of the argument

Before establishing the model formally in a lattice framework, we present the key ideas informally in a continuous time setting. Interest rates are constant. There is a single risky asset that pays no dividends. We work throughout with discounted prices; the price of the asset at time t is denoted by S_t . At time 0, a bank sells an American option to an investor. The investor can choose to exercise the option at any time t before maturity T . On exercise, the holder receives a discounted sum, $A(S_t, t)$. There is a complete set of European options (all strikes and maturities) available.

A general portfolio of European options that expire at or before T is characterized by the function V with $V(S, T) = 0$; the interpretation of V is that the portfolio generates cash at the rate $-\partial V / \partial t$ (so the portfolio generates $V(S_t, t)$ from time t onwards if the asset price remains constant until the option expires; in this example we assume that V is differentiable).

The first proposition below states that, if the function V is convex in S , decreasing in t and greater than A (the pay-out to the American option) throughout the domain, then the European portfolio V dominates A . The argument is based on arbitrage. Suppose at time 0 the agent is long the European portfolio and short the American option. The agent does nothing until time t^* when the option is exercised; she then delta hedges the European

portfolio by going long $\partial V/\partial S$ units of the underlying. The cash flow from time 0 to maturity is:

$$\begin{aligned}
 (1) \quad & \left\{ -\int_0^{t^*} \frac{\partial V}{\partial t} dt \right\} - A(S_{t^*}, t) - \left\{ \int_{t^*}^T \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS \right\} \\
 & \geq -A(S_{t^*}, t^*) - \left\{ \int_{t^*}^T dV \right\} && \text{(since } \frac{\partial V}{\partial t} \leq 0 \text{ and } V \text{ is convex)} \\
 & = -A(S_{t^*}, t^*) + V(S_{t^*}, t^*) && \text{(since } V(., T) = 0) \\
 & \geq 0 && \text{(since } V \geq A).
 \end{aligned}$$

The price of the European portfolio is an upper bound on the price of the American option. The second proposition then states that the cheapest such portfolio provides the lowest possible bound on the price of the American option.

Consider for example the case of an American put option with strike K . The function A is:

$$\begin{aligned}
 (2) \quad & A(S, t) = \max \{ K_t - S, 0 \}, \\
 & \text{where } K_t = Ke^{-rt}.
 \end{aligned}$$

$V = A$ satisfies the requirements of proposition 1. It corresponds to a portfolio that pays at rate rK_t while $S_t \leq K_t$, and 0 otherwise; and also pays $\max\{K_T - S_T, 0\}$ at maturity. The hedging strategy is to do nothing until the option is exercised at time t^* . The bank then borrows K_{t^*} , buys one unit of the asset and pays the investor $K_{t^*} - S_{t^*}$. While the asset remains below the strike price, the cash flow from the portfolio pays the interest on the debt, so the discounted value of the debt is always K_t . At maturity, the portfolio pays $K_T - S_T$, so the bank can sell the asset and repay the debt. If at some time $t > t^*$ the option goes out of the money, so $S_t > K_t$, the asset is sold, the debt is repaid, and the bank ends up with a positive cash position.

We will show that this is sometimes, but not always, the cheapest dominating portfolio. Having illustrated the approach, we now present the formal model.

2.2 The formal set up

The model is set in a discrete time, discrete space framework. The time index t goes from 0 to T . The interest rate is deterministic. Prices are discounted at the riskless rate to time

0. There is a single risky security. It pays no dividends¹ and is traded in a frictionless market at (discounted) price S_t . S_t can take one of J possible values:

$$(3) \quad S_t \in \{x_j \mid j = 1, \dots, J\} \text{ for all } t, \text{ with } x_1 < \dots < x_J.$$

For convenience, the price nodes are spaced geometrically with $x_{j+1} = ux_j$, and the time periods each of length δ .

An American option is characterized by a $J \times T$ *pay-off matrix* $\mathbf{A} = \{a_{j,t}\}$. If the option is exercised at time t , the holder receives a pay-off whose discounted value is $a_{j,t}$ ² where j is given by $S_t = x_j$. Once the option has been exercised, it is dead; it cannot be exercised again. The holder can chose to let the option expire without exercising it.

For each node (j,t) there is a traded security that pays 1 if $S_t = x_j$ and 0 otherwise³; its price at time zero is $p_{j,t}$. The $J \times T$ matrix $\{p_{j,t}\}$ is denoted by \mathbf{P} . $p_{j,t}$ can be interpreted as the risk-adjusted probability of state j occurring at time t , given information at time 0. The absence of arbitrage implies the existence of a process under which S is a martingale and $\Pr\{S_t = x_j\} = p_{j,t}$ for all (j, t) (Davis and Hobson 2007).

2.3 Dominating portfolios

A portfolio of elementary securities is represented by $\mathbf{E} = \{e_{j,t}\}$. The portfolio pays the holder an amount $e_{j,t}$ at time t if $S_t = x_j$. The cost of the portfolio, $c(\mathbf{E})$, is the cost of the elementary securities that compose it:

¹ The extension to assets with a constant yield, or a known dividend, is described in 2.4.

² It is implicitly assumed that the American option cannot be exercised at time 0; this is for notational convenience and has no effect on the results.

³ We could instead have assumed the existence of a complete set of conventional European call options (that is an option for each strike x_j and each maturity t). The Arrow-Debreu claims we are assuming can be created from the call options using a butterfly spread – a long position in calls with strikes x_{j-1} and x_{j+1} and a short position in calls with strike x_j .

$$(4) \quad c(\mathbf{E}) = \sum_{j,t} e_{j,t} p_{j,t}.$$

The following proposition sets out sufficient conditions under which the European portfolio \mathbf{E} dominates the American option \mathbf{A} .

PROPOSITION 1: *if there exist three $J \times T$ matrices \mathbf{E} , \mathbf{D} and \mathbf{V} that satisfy the following conditions:*

$$(5) \quad \begin{aligned} &1) \quad e_{j,t} \geq 0 \text{ for all } j, t; \\ &2) \quad v_{j,t} \leq e_{j,t} + d_{j,t} (x_m - x_j) + v_{m,t+1} \text{ for all } j, m, t, \\ &\quad \text{where } v_{m,T+1} \equiv 0; \\ &3) \quad v_{j,t} \geq a_{j,t} \text{ for all } j, t, \end{aligned}$$

then the cost of \mathbf{E} is an upper bound on the price of the American claim \mathbf{A} .

Proof: suppose the proposition is false. An agent writes the American option at time 0, and uses the proceeds to buy the European option \mathbf{E} . The agent does nothing until time t^* when the American option is exercised. At time $t \geq t^*$, the agent hedges by holding d_{jt} of the underlying asset, where j is given by $S_t = x_j$.

By assumption, \mathbf{A} costs more than \mathbf{E} , so the cash flow at inception is strictly positive. At every period up to time t^* , the agent receives an amount e_{jt} which is positive by condition (1). Condition (2) ensures that the proceeds from the delta hedged portfolio \mathbf{E} from time t^* to time T aggregate to at least v_{j^*,t^*} . Condition (3) ensures that this exceeds a_{j^*,t^*} , the exercise cost of the American option. The aggregate cash flow is also strictly positive if the American option is never exercised. This is an arbitrage strategy. To avoid arbitrage, the price of the American option must be less than or equal to the price of the European portfolio. ■

v_{jt} can be regarded as the *intrinsic value* of the European portfolio \mathbf{E} at node (j, t) . By delta hedging the portfolio (represented by the matrix \mathbf{D}), the holder can ensure that the hedged cash flow from time t onwards will equal at least $v_{j,t}$ whatever path the asset subsequently follows. Proposition 1 can be restated as: if the portfolio of European

options \mathbf{E} has positive cash flows only, and if the intrinsic value of \mathbf{E} exceeds the immediate exercise value of the American option in all future states, then \mathbf{E} dominates the American option.

The proposition identifies a set of strategies that bound the price of an American option. To find the tightest such bound, the problem can be written as a linear program:

$$\begin{aligned}
 &\mathbf{LP1}: \text{Find the triplet of } J \times T \text{ matrices } \{\mathbf{D}, \mathbf{E}, \mathbf{V}\} \\
 &\quad \text{that minimizes } \sum_{j,t} e_{j,t} p_{j,t} \\
 &\quad \text{subject to the constraints:} \\
 &1) \quad e_{j,t} \geq 0 \text{ for all } j, t; \\
 &2) \quad v_{j,t} \geq a_{j,t} \text{ for all } j, t; \\
 &3) \quad e_{j,t} + (x_m - x_j) d_{j,t} - v_{j,t} + v_{m,t+1} \geq 0 \text{ for all } j, m, t \\
 &\quad \text{where } v_{m,t} \equiv 0 \text{ when } t = T + 1.
 \end{aligned}$$

The feasible set is not empty⁴. The solution is bounded below by zero; otherwise there would exist a portfolio \mathbf{E} that has positive cash flows in all states of the world, and has a negative cost, and this would be an arbitrage. So LP1 has a solution; denote the solution with an asterisk. Proposition 1 implies that the price of the American option is bounded above by $c(\mathbf{E}^*)$. The next step is to show that this is the least upper bound.

2.4 Feasible prices

A process for the risky asset is *consistent* if it is martingale and if the probability of reaching any node (j, t) is $p_{j,t}$. The expected pay-off to a contingent claim under a consistent process is a *feasible* price in the sense that introducing that claim into the market at that price cannot lead to arbitrage. We search among a carefully chosen class of processes for that which maximizes the expected pay-off. We show that this search is the dual of LP1, and so has the same solution, $c(\mathbf{E}^*)$. It is both an upper bound and a feasible price. It must therefore be the supremum.

⁴ For example, define $e_{j,t} = v_{j,t} = \max[a_{j,t}, 0]$ and $d_{j,t} = 0$ for all j and t .

We model the process as a regime-switching Markov chain. There are two regimes. Initially, the model is in regime 1. It may switch to regime 2 at some future date; if it does so, it stays in regime 2. There are J price nodes. At $t=1$, the probability of being at node j is $p_{j,1}$ (given by the price of 1-period options at time 0). The chain is characterized by specifying the *joint* probability of successive states (not the *conditional* probability):

$\mathbf{F} = \{f_{j,t}\}$ is the probability of being at node j at time t and switching from regime 1 to regime 2 at that node;

$\mathbf{G} = \{g_{j,m,t}\}$ is the probability of being at node j at time t in regime 1, and moving to node m next period;

$\mathbf{H} = \{h_{j,m,t}\}$ is the probability of being at node j at time t in regime 2 and moving to node m next period.

If the model is in state j in period t then $S_t = x_j$. Conditional probabilities can be readily computed; for example the probability of the price moving from x_j at time t to x_k the following period under regime 1 is $g_{j,k,t} / \sum_m g_{j,m,t}$.

The following conditions are imposed on $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$:

$$\begin{aligned}
 & 1) \ f_{j,t} \geq 0; \quad 2) \ g_{j,t} \geq 0; \quad 3) \ h_{j,m,t} \geq 0; \\
 & 4) \ \sum_m (x_m - x_j) g_{j,m,t} = 0; \quad 5) \ \sum_m (x_m - x_j) h_{j,m,t} = 0; \\
 (6) \quad & 6) \ \sum_m (g_{j,m,t} + h_{j,m,t}) = p_{j,t}; \quad 7) \ \sum_m (g_{m,j,t} + h_{m,j,t}) = p_{j,t+1} \ (t < T); \\
 & 8) \ f_{j,t} = \sum_m (h_{j,m,t} - h_{m,j,t-1}) \quad (\text{with } h_{m,j,t} \equiv 0 \text{ when } t = 0); \\
 & \text{for } j = 1, \dots, J \text{ and } t = 1, \dots, T.
 \end{aligned}$$

Conditions (1-3) ensure that the probabilities are positive; conditions (4-5) ensure that S_t is a martingale under both regimes; conditions (6-7) ensure that the probability of entering a node and the probability of leaving the node are both equal to the probability of being at that node; condition (8) says that the probability of switching regime at a node

is equal to the difference between the probability of arriving at the node in regime 2 and the probability of leaving it in regime 2.

The conditions are both necessary and sufficient to ensure that the process is well-defined, that S is martingale, and that the probability distribution of S_t is consistent with the initial risk-neutral density \mathbf{P} . Now suppose that the holder of the American option decides to exercise it when the regime switches. Under the process $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\}$, the value of the American option is:

$$(7) \quad v(\mathbf{F}) = \sum_{j,t} f_{j,t} a_{j,t}.$$

$v(\mathbf{F})$ is the value of the American option in a consistent model with a given exercise strategy. There is no reason to believe the strategy is optimal, but $v(\mathbf{F})$ must represent at least a lower bound on the feasible price in this model.

Consider the linear program:

LP2: Find the $J \times T$ matrix \mathbf{F} and the two $J \times J \times T$ matrices \mathbf{G} and \mathbf{H} that maximizes $\sum_{j,t} f_{j,t} a_{j,t}$ subject to the constraints in (6).

The feasible set is not empty; consider for example the strategy of never exercising the American claim, so $\mathbf{F} = \mathbf{H} = \mathbf{0}$. \mathbf{G} is then just a martingale process for the underlying asset that is consistent with the prices of the European options; such a process must exist if the market is free of arbitrage. The problem is also bounded since \mathbf{F} is bounded ($p_{j,t} \geq f_{j,t} \geq 0$). So there is a solution $v(\mathbf{F}^*)$.

PROPOSITION 2: *the search for the cheapest dominating portfolio (LP1) is equivalent to the dual of the search for the process that maximizes the value of the American claim (LP2). The cost of the cheapest dominating portfolio $c(\mathbf{E}^*)$ is equal to the maximum feasible price $v(\mathbf{F}^*)$, and they are therefore both equal to the least upper bound on the value of the American claim.*

Proof: the proof involves re-writing the two linear programs in standard form and showing that they are dual. It is set out in the Appendix. ■

2.5 Extending the model

Propositions 1 and 2 apply to an American option on an asset that pays no dividends. They can readily be extended to assets that do pay dividends, provided that the present value of the dividend is a known function of the asset price. For example, take the case where the option is on a share that has a dividend yield of d per period. For the purposes of the model, the underlying asset is a share with dividends reinvested. By construction, this notional underlying asset pays no dividends. At node (j, t) the value of the underlying is x_j , the value of one share is $x_j / (1+d)^t$, and the immediate exercise value of an American call with nominal strike K is:

$$(8) \quad a_{j,t} = \frac{x_j}{(1+d)^t} - \frac{K}{(1+i)^t}$$

where i is the per period interest rate. The extension to discrete proportional dividends, and also to known cash dividends is straightforward, as is the extension to time varying but non-stochastic interest rates and dividend yields.

The model can also be used to bound the price of Bermudan-style options where the holder's right to exercise the option is restricted to particular times or periods. If the holder cannot exercise the option at time t then a_{jt} is set equal to zero for all j . Since the holder of a live option can choose to allow the option to expire unexercised, it is rational to defer exercise when $a = 0$ and so keep the option alive.

3. EXPLORING THE BOUNDS

3.1 Bounds on the American put

We illustrate the model by looking at the bounds on American put options in the presence of European options whose prices are set by a trinomial process that approximates a diffusion with constant volatility σ . The model is in discrete time, with T equal periods

starting at 0 and ending at τ . The length of each period is $\delta = \tau/T$. The American option can be exercised at any node; as δ goes to zero, the permissible exercise dates converge to the continuum $[0, \tau]$.

Price nodes are distributed geometrically with $x_{j+1} = ux_j$ ($u > 1$). European option prices are equal to their expected pay-offs when the discounted price of the risky asset follows the trinomial process:

$$(9) \quad S_{t+1} = \begin{cases} S_t u & \text{with probability } \lambda/(1+u) \\ S_t / u & \text{with probability } u\lambda/(1+u) \\ S_t & \text{with probability } 1-\lambda \end{cases}$$

where $\lambda \in (0,1)$ and $\sigma^2 = \lambda (\log u)^2 / \delta$.

In the following examples, the initial asset price S_0 is 100, the time to maturity τ is 1 year, and the annualized volatility σ is 10%. There are 50 periods per year. The probability of a price change at each node, λ , is set equal to 2/3, and $u = 1.0175$.

Table I sets out the upper bound on the price of the American put option for different values of the strike price K and the annual (continuously compounded) interest rate r . The bound is compared with the conventional valuation of an American option – the expected pay-off to the option under the process in (9) under the optimal exercise strategy. Two European put option valuations are also reported. One is the value of a European put option $P^E(K, T)$ with strike K and that matures when the American option expires. The other is the value of the European put option with strike K and maturity of T or less that has maximum value, $\text{Max}[P^E(K, t) | t \leq T]$, denoted by $P^E(K, \leq T)$. We define the American premium as the difference between the American put price and $P^E(K, \leq T)$. It represents the premium the holder pays for being able to specify the exercise time later, rather than having to choose the exercise time at inception.

The value of an option depends on the level of the strike relative to the spot. Two measures of the moneyness of the option are reported, at inception, and at expiry:

$$(10) \quad \begin{aligned} d_0 &= \ln(S/K)/\sigma\sqrt{\tau}; \\ d_T &= \ln(S/Ke^{-r\tau})/\sigma\sqrt{\tau}. \end{aligned}$$

Table I: Upper Bound on the Value of an American Put

Strike (K)	American Bound	Diffusion	European $P^E(K, \leq T)$	European $P^E(K, T)$	American premium (diffusion/bound)	Moneyness d_0 d_T	
	(1)	(2)	(3)	(4)	(5) = (2-3)/(1-3)	(6)	(7)
<i>r = 2%</i>							
95	1.67	1.42	1.35	1.35	20%	0.51	0.71
100	3.76	3.22	3.03	3.03	26%	0.00	0.20
105	6.89	6.13	5.68	5.68	37%	-0.49	-0.29
110	10.84	10.12	10.00	9.17	15%	-0.95	-0.75
<i>r = 5%</i>							
95	1.34	0.91	0.77	0.77	25%	0.51	1.01
100	3.44	2.43	1.94	1.94	33%	0.00	0.50
105	6.63	5.33	5.00	3.92	20%	-0.49	0.01
110	10.68	10.00	10.00	6.82	0%	-0.95	-0.45
<i>r = 10%</i>							
95	0.83	0.44	0.26	0.26	31%	0.51	1.51
100	2.90	1.61	1.01	0.79	32%	0.00	1.00
105	6.25	5.00	5.00	1.90	0%	-0.49	0.51
110	10.46	10.00	10.00	3.76	0%	-0.95	0.05

The table shows the upper bound on the value of an American put with maturity of 1 year for various levels of strike and interest rates, when the spot price of the underlying is 100, and European options are trading on an implied annual volatility of 10%. The second column shows the price of the American put assuming the asset has constant volatility; the next two columns show the prices of European options with the same strike and maturities of either up to 1 year, or of 1 year exactly. Column 5 shows the ratio of the American option premium under the diffusion process to the maximum premium. d_0 and d_T are measures of moneyness, defined in the body of the paper. Values are calculated using 40 steps/year.

One striking feature of table I is how small is the ratio (shown in column 5) between the value of being American assuming that the process is a diffusion, and the upper bound on the value of being American. Taking a crude average across strikes and interest rates, the standard valuation of model value is only 20% of its rational bound, and for no strike or interest rate does it rise above 37%, suggesting that the American feature may be seriously undervalued by standard models.

With five parameters in the model (S, K, r, τ, σ) , table I appears to sample only a small part of the parameter space. But relative prices are largely determined by the moneyness vector (d_0, d_T) ; holding the moneyness constant, the option prices, whether American or European, are approximately proportional to $S\sigma\sqrt{\tau}$ ⁵, so the percentage premium depends mainly on the (two-dimensional) moneyness vector. The range of parameters in Table I is chosen to give a suitably wide range of values for the moneyness vector.

To exhibit the bounding European portfolios, the intrinsic value $v_{j,t}$ of the portfolio is plotted as a function of x_j for different values of t . The lines are convex; it follows that the intrinsic value $v_{j,t}$ is identical to the sum $\sum_{s=t}^T e_{j,s}$. The values of e , the pay-out from the portfolio in each period, can be read off by looking at the difference between successive curves. Figure 1 shows these portfolios for an out of the money option ($K = 98$) and an in the money option ($K = 102$). In both cases, using Proposition 1, it is easy to verify that the portfolios do dominate the American put.

For the out of the money American put, the dominating portfolio takes a particularly simple form:

$$(11) \quad \begin{aligned} v_{j,t} &= \left[K_t - x_j \right]^+; \\ e_{j,t} &= v_{j,t} - v_{j,t+1}. \end{aligned}$$

This portfolio is the same as that given in the informal presentation in section 2.1. While this portfolio always dominates the American put, the lower panel of Figure 1 shows that it is not the cheapest dominating portfolio for all values of strike.

⁵ If the discounted exercise price K_t were linear in time, and if the process generating European option prices had constant variance of price changes, dS_t then, holding moneyness constant, prices of all options (European, American, upper bound) would be exactly proportional to the standard deviation of the terminal price. In our model, it is $\ln K_t$ that is linear in time, and $\ln(dS_t)$ that has constant variance, but for low interest rates and volatility (small values of $r\tau$ and $\sigma\sqrt{\tau}$) the homogeneity property holds approximately.

3.2 Extreme Processes

The dominating portfolios give insight into the features of price processes that make American options particularly valuable. Suppose a bank writes an out of the money American put, is long the dominating portfolio, and has zero cash. The bank does nothing until the put is exercised at some time t^* . If the bank then delta hedges the European portfolio until maturity, it will extract its intrinsic value, and end with a net positive cash position, as already shown. But if the European options are traded, there is an alternative strategy open to the bank: when the customer exercises the American option, the bank can liquidate its own portfolio of European options and use the proceeds to pay its customer. If there are no arbitrage opportunities, the bank will be able to sell its European hedge portfolio for at least its intrinsic value, and so still end with positive cash.

A bank following this second strategy will have *strictly* positive cash if the American option is still unexercised in any period when the European portfolio is cash generating, or if, at the point of exercise, the European portfolio is worth more than its intrinsic value. Processes that maximize the value of the American option, holding the prices of European options constant, are those that leave the bank with zero cash. These processes have the property that, in the early exercise region, the European portfolio trades only at intrinsic value, so its implied volatility is zero. This implies that, conditional on the point of optimal exercise having been reached (“dead paths”), the volatility of the path should be low⁶.

The initial prices of European options determine the forward implied volatility, and hence the expected future volatility conditional on stock price and time (Britten-Jones and Neuberger 2000). If the volatility of dead paths is low, then the volatility of live paths going through the same nodes on the lattice would have to be high.

⁶ Low, but not necessarily zero. For example, the implied volatility at time t of an option with strike K ($> S_t$) and maturity T will be zero provided that the asset price will, with probability 1, remain below K until maturity.

The type of process that supports a high American premium is one where there is a major source of uncertainty that will be resolved at an unknown time within a fixed period, and the process of resolution is likely to occur over a short period. For then optimal exercise will wait until the uncertainty is resolved, and at that time implied volatilities will be very low. Set in the context of the standard types of process used for modeling asset prices (stochastic volatility, GARCH, regime switching) this type of process is anomalous in having volatility being *negatively* correlated over time. However such behavior is not wholly implausible in reality. For example the market may believe that within the next six months there could well be a bid for a particular company, but the timing is uncertain. The size of any bid may be uncertain, but once a bid is made, much of the uncertainty would vanish. In currency markets, a fixed exchange rate that is liable to speculative attack may similarly exhibit a spike of volatility at some unknown time followed by a period of calm. In such markets, an American option would be particularly valuable relative to a European option.

To make this insight more concrete, we look at a simple (if rather artificial) process in which the value of an American option approaches its upper bound. To keep consistency with the previous Brownian example, the process is designed to deliver broadly similar European option values.

The Single Jump Process (SJP) for S_t is defined by:

$$(12) \quad S_t = \begin{cases} S_0 & t < t^* \\ S_0 e^{\sigma \varepsilon - \sigma^2/2} & t \geq t^* \end{cases}$$

where ε is a standard normal variate and t^* is uniformly distributed on $[0,1]$.

S_t has a lognormally distributed single jump at a random time t^* . The distribution of the asset price at one year is lognormal, with variance parameter σ^2 , so European option prices with maturity of 1 are identical to Black-Scholes prices. As with standard

Brownian motion, the variance of returns increases almost linearly with horizon⁷, but the distribution prior to maturity is not lognormal.

Under the SJP, the optimal exercise strategy for an out-of-the-money American put option is to exercise immediately after the jump, if at all. The dominating strategy in (11) exactly replicates the pay-off to the American option in all states of the world. The SJP maximizes the value of the American premium under all possible processes consistent with the same set of European option prices.

Table 2 shows the pricing of American and European options for the same interest rates and maturities as Table 1. For out of the money options, the American option price is at its rational bound while under the Markov diffusion process⁸ that supports the same European option prices, the American option premium is under half the rational bounds premium. For in the money options, the option value is well below rational bounds, but is generally substantially higher than it would be under a Markovian diffusion process.

⁷ Strictly, the variance of returns over the period $[0,t]$ is $\exp(\sigma^2 t) - 1$ under Brownian motion and $t(\exp(\sigma^2) - 1)$ under the SJP.

⁸ The lognormal is approximated as a 41 node logbinomial. The Markov diffusion is approximated as a trinomial on the same prices. Since the local implied volatility is unbounded, we use an ever finer time step as t approaches zero.

Table 2: American Option Bounds and Prices with a Single Jump Process

Strike (K)	American			European		American premium	
	Bound	Jump	Diffusion	$P^E(K, \leq T)$	$P^E(K, T)$	(jump/bound)	(diffusion/bound)
	(1)	(2)	(3)	(4)	(5)	(6)=(2-4)/(1-4)	(7) = (3-4)/(1-4)
<i>$r = 2\%$</i>							
95	1.63	1.63	1.46	1.38	1.38	100%	34%
100	3.51	3.51	3.24	3.08	3.08	100%	36%
105	6.76	6.34	5.99	5.71	5.71	60%	26%
110	10.81	10.03	10.00	10.00	9.20	4%	0%
<i>$r = 5\%$</i>							
95	1.27	1.27	1.00	0.78	0.78	100%	47%
100	2.87	2.87	2.34	1.92	1.92	100%	45%
105	6.41	5.40	5.00	5.00	3.89	28%	0%
110	10.65	10.00	10.00	10.00	6.76	0%	0%
<i>$r = 10\%$</i>							
95	0.87	0.87	0.60	0.39	0.26	100%	45%
100	2.08	2.08	1.48	1.00	0.79	100%	44%
105	6.01	5.00	5.00	5.00	1.88	0%	0%
110	10.48	10.00	10.00	10.00	3.74	0%	0%

European option prices are set as if the underlying follows the Single Jump Process. Of the three American option prices, the first is the rational bound, the second is the price under the pure jump process while the third is the price under the Markov diffusion process that is consistent with the European option prices.

In the first part of this section we showed that the American option premium is a small fraction of its upper bound when the underlying process is a constant volatility diffusion. Similar computations (not reported here) using other widely used processes (jump-diffusion, stochastic volatility) with plausible parameters support the same conclusion. We then showed that it is possible to get high American premiums, but this necessitates some kind of two state process where volatility falls sharply at some random time. The question naturally arises whether the bounds can be tightened significantly by placing modest restrictions on the process.

4. TIGHTENING THE BOUNDS

In this section we explore the possibility of generating tighter but still robust bounds by excluding some implausible behavior. The “good-deal” literature starting with Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000) is motivated by a similar desire to find restrictions on pricing in incomplete markets that are tighter than no arbitrage.

They assume that the objective process for the underlying is known, and impose restrictions on the behavior of the stochastic discount factor. In the context of our model, the natural way to exclude extreme behavior is to place restrictions on the future prices of European options. In particular, we restrict the implied volatility on which options will trade in future.

The empirical justification for this is that while implied volatilities do vary widely over time, they do not appear to approach zero. For example, the VIX index, which is traded on the Chicago Board Options Exchange, measures the implied volatility of an at-the-money option on the S&P 500 Index with 30 days to expiry. Since its introduction in 1993 to March 2009, it has averaged 20% and reached a minimum of 8.6%. Inspection of the time series of the VIX index in Figure 2 suggests that it should be possible to nominate some non-trivial floor level for the implied volatility of a European portfolio over its lifetime without a large sacrifice of robustness.

Avellaneda, Levy, and Paras (1995) examine the impact of bounds on the actual volatility of the underlying price process on option prices. They assume that the asset follows a diffusion, and in effect they constrain the quadratic variation of the price path. This excludes both jumps and periods of price stability. By contrast, we do not exclude any price process, but instead impose restrictions on the prices of contingent claims that are likely to be satisfied.

To implement the idea, consider a “local volatility” contract that pays \$1 if $S_{t+1} \neq S_t$, and zero if the price is unchanged. The more volatile the asset, the higher the price of the local volatility contract. Now assume that, for each node (j,t) , it is known that the price of the local volatility contract purchased at date t will not fall below some level $\lambda_{j,t}$. With the minimum total return on a price change in the being of size u this corresponds to a floor on the implied volatility of the contract at this node of $\sqrt{\lambda_{j,t}}/\delta \ln(u)$, where δ is the length of the time period.

This restriction is readily incorporated in the linear program. Recall the matrix $\mathbf{H} = \{h_{jmt}\}$, which represents the probability of being at node j at time t , at node m at time $t+1$ and the option having been exercised by time t . The minimum volatility constraint is:

$$(13) \quad h_{j,j,t} \leq (1 - \lambda_{j,t}) \sum_m h_{j,m,t} \quad \text{for all } j, m, t,$$

This modifies LP1 and produces the new problem:

LP3: Find the four $J \times T$ matrices $\{\mathbf{D}, \mathbf{E}, \mathbf{V}, \mathbf{Y}\}$

that minimizes $\sum_{j,t} e_{j,t} p_{j,t}$

subject to the constraints:

- 1) $e_{j,t} \geq 0$ for all j, t ;
- 2) $v_{j,t} \geq a_{j,t}$ for all j, t ;
- 3) $e_{j,t} + (x_m - x_j) d_{j,t} + (\lambda_{j,t} - I_{m \neq j}) y_{j,t} - v_{j,t} + v_{m,t+1} \geq 0$ for all j, m, t
where $v_{m,t} \equiv 0$ when $t = T + 1$ and $I_{m \neq j} = 1$ if $m \neq j$, and 0 otherwise;
- 4) $y_{j,t} \geq 0$ for all j, t .

The interpretation of constraint (3) is that in addition to delta hedging by going long $d_{j,t}$ units of the underlying, the agent can write $y_{j,t}$ (≥ 0 by virtue of constraint 4) local volatility contracts. The constraint exploits the assumption that the price of such a contract will be at least $\lambda_{j,t}$.

Table III shows that imposing a floor on implied volatility tightens the bounds substantially. With a floor on the future implied volatility of options set at half the current level, the bound on the American premium is reduced by around 20% on average, with the effect being most pronounced for low strike options. As the floor on future implied volatility approaches 10%, which is today's forward volatility, so uncertainty about future volatility is constrained, and the bound approaches the diffusion price.

Table III: Bound on the American Put with a Floor Level of Implied Volatility

<i>Strike (K)</i>	<i>Bound on American, with floor on implied vol of</i>			<i>American, Diffusion</i>	<i>European $P^E(K, \leq T)$</i>	<i>American option premium, relative to robust bound</i>		
	<i>0%</i>	<i>5%</i>	<i>8%</i>			<i>5% floor</i>	<i>8% floor</i>	<i>Diffusion</i>
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
<i>r = 2%</i>								
95	1.67	1.52	1.44	1.42	1.35	52%	29%	20%
100	3.76	3.51	3.30	3.22	3.03	66%	37%	26%
105	6.89	6.74	6.33	6.13	5.68	87%	54%	37%
110	10.84	10.78	10.51	10.12	10.00	94%	60%	15%
<i>r = 5%</i>								
95	1.34	1.13	0.98	0.91	0.77	63%	37%	25%
100	3.44	3.11	2.65	2.43	1.94	79%	48%	33%
105	6.63	6.49	5.92	5.33	5.00	92%	57%	20%
110	10.68	10.63	10.40	10.00	10.00	92%	59%	0%
<i>r = 10%</i>								
95	0.83	0.67	0.51	0.44	0.26	72%	45%	31%
100	2.90	2.62	1.97	1.61	1.01	85%	51%	32%
105	6.25	6.13	5.69	5.00	5.00	90%	55%	0%
110	10.46	10.42	10.27	10.00	10.00	91%	58%	0%

The table shows the upper bound on the value of an American put with maturity of 1 year for various levels of strike and interest rates, when the spot price of the underlying is 100, and European options are trading on an implied annual volatility of 10%. The first column is the unrestricted upper bound, while the second and third assume that European options will never trade in future on an implied volatility of less than 5% and 8% respectively. Column four shows the price of the American put assuming the asset follows the diffusion process; and column five shows the price of the most valuable European option with the same strike and maturity of up to 1 year. The next three columns show the American option premiums implicit in columns 2-4 as a proportion of the premium implicit in the ration bound, column 1. Values are calculated using 40 periods.

In the limit, the lower bound on future volatility could be set equal to today's forward volatility; as shown formally below, this corresponds to the hedging strategy of Derman, Ergener and Kani (1995). If the floor on implied volatility is set higher than this, the model ceases to be arbitrage free; agents can buy volatility through buying options at time zero, and subsequently sell it back using local volatility contracts.

For each node define the jump probability:

$$\lambda_{j,t}^F \equiv \begin{cases} 0 & \text{if } c_{j,t+1} = c_{jt} \text{ and } p_{j,t} = 0; \\ \infty & \text{if } c_{j,t+1} \neq c_{jt} \text{ and } p_{j,t} = 0; \\ \frac{x_{j+1} - x_{j-1}}{(x_{j+1} - x_j)(x_j - x_{j-1})} \frac{c_{j,t+1} - c_{j,t}}{P_{j,t}} & \text{otherwise;} \end{cases}$$

(14) where $p_{j,t}$ is the price of a security that pays \$1 if $S_t = x_j$,
and $c_{j,t}$ is the time zero price of a call option with strike x_j and maturity t .

LEMMA

If $\lambda_{j,t}^F \leq 1$ for all j and t , there exists a martingale process Q for S_t where:

$$(15) \quad \Pr\{S_{t+1} = x_k | S_t = x_j\} = \begin{cases} 1 - \lambda_{j,t}^F & \text{if } k = j; \\ \lambda_{j,t}^F \frac{x_{j+1} - x_j}{x_{j+1} - x_{j-1}} & \text{if } k = j-1; \\ \lambda_{j,t}^F \frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}} & \text{if } k = j+1; \\ 0 & \text{otherwise.} \end{cases}$$

under which $\Pr\{S_t = x_j\} = p_{j,t}$ for all j, t .

PROOF: this result is due to Dupire (1994), and Derman and Kani (1994). The process is well-defined provided that $\lambda_{j,t}^F \in [0, 1]$. $\lambda^F \leq 1$ by assumption; $\lambda^F \geq 0$ since, in (14), the absence of arbitrage ensures that $p_{j,t} \geq 0$ and $c_{j,t+1} \geq c_{jt}$. The process is clearly martingale. The consistency with the $\{p_{j,t}\}$ can be proved by induction: if $\Pr\{S_u = x_j\} = p_{j,u}$ for all j and all $u < t$, then substitution of (14) into (15) gives $\Pr\{S_u = x_j\} = p_{j,u}$ for $u = t$.

PROPOSITION 3:

1. if, for some j and t , $\lambda_{j,t} > \lambda_{j,t}^F$, there is an arbitrage;
2. if, for all j and t , $\lambda_{j,t} \leq \lambda_{j,t}^F \leq 1$, there is no arbitrage;

3. if, for all j and t , $\lambda_{j,t} = \lambda_{j,t}^F \leq 1$, and if the immediate exercise value of the American option is a convex function of the stock price, then the cost of the robust hedge is the same as the valuation of the American option under the diffusion Q . The robust hedge is a portfolio that pays at each node the difference between the value of the American option at that node and its continuation value.

PROOF: the formal proof is in the Appendix. The first part is demonstrated by buying a forward volatility contract at time zero, and selling a volatility contract at time t . The second follows directly from the Lemma: since there is a martingale process that is consistent with all the restrictions, there is no arbitrage. The last part is due to the fact that there is just one process that is consistent with all the restrictions, so the upper bound on the American option is simply its value under that process.

The robust hedge referred in the third part of Proposition 3 is very similar to the static hedging strategy of Derman, Ergener and Kani (1995; hereafter DEK). They show how to set up a portfolio of European options that, in a Black-Scholes world has the same value function as the exotic that is to be hedged; they develop the idea in the context of using European options to hedge barrier options, but it applies equally to American options.

5. RATIONAL BOUNDS AND ROBUST HEDGES

A valuable by-product of the search for option pricing bounds is the discovery of robust hedging strategies that enforce those bounds. We have shown how to create a spectrum of robust hedging strategies for American options, ranging from the entirely robust but expensive rational bounds hedge at one extreme to the much cheaper DEK strategy. The cheaper the strategy the more exposed is the hedger to the risk that future implied volatility will differ from today's forward volatility. The balance between robustness and price depends on the preferences of the hedger. In this section, we use simulation to explore the distribution of the hedging error with robust hedges and with more conventional delta-gamma hedges.

The experiment is set up as follows: the underlying asset price follows a Heston (1993) stochastic volatility process. At time zero, the bank writes an American put option, and establishes a hedge. The hedge is liquidated when the option is exercised or expires. The American option and the European options used for hedging all trade at their fair prices, and the American option is exercised whenever it is optimal to do so. We consider different hedging strategies followed by the bank. Under **robust hedging** the bank sets up the robust hedge at the time it writes the option and liquidates the hedge when the option is exercised. We consider both the unrestricted robust hedge that enforces the rational bounds, and the restricted version described in the previous section where there is a floor on implied volatility. Under **delta-gamma hedging**, the bank hedges dynamically from inception to exercise using two securities: the underlying asset, and the European option with the same strike and maturity as the American option being hedged.

If the bank knew the true process, then it could hedge perfectly since there are no frictions, and there are two hedge instruments to hedge two sources of uncertainty. But the bank mistakenly assumes that the world is Black-Scholes. The bank observes the price of the hedging option at each point in time, computes its implied volatility using the Black-Scholes model, and uses that volatility (and the Black-Scholes model) to determine the hedge ratios. Each period the hedge portfolio is rebalanced to match the first and second derivatives of portfolio value with asset price (under the Black-Scholes model) to that of the American option.

The price processes are simulated assuming the risk premia are zero, so the discounted prices of the asset and of the options follow martingales. With all transactions taking place at fair price, the mean profit is zero whatever hedging strategy is used. We run 100,000 simulations of both models and report the standard deviation and the 99th percentile.

The Heston process is:

$$(16) \quad \begin{aligned} dS_t/S_t &= \sqrt{v_t} d\tilde{z}_t^s; & dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} d\tilde{z}_t^v; \\ E[d\tilde{z}_t^s \cdot d\tilde{z}_t^v] &= \rho dt. \end{aligned}$$

We implement it on a three dimensional lattice (asset price, volatility level and time); each period, the state vector (S_t, v_t) can move to one of six possible nodes (v can go up or down, while S can go up, down or remain the same). The Black-Scholes implied volatility of an option is computed as the (constant) jump probability λ that generates the market value of the option. Deltas and gammas are computed using the values of the option one period ahead⁹.

To retain comparability with our previous results, the mean of the variance of returns, θ , is 0.01. The volatility mean reversion parameter κ is 2 (on an annual basis – so volatility mean reverts with a time constant of 6 months). The volatility of volatility σ is 0.2, so the standard deviation of the instantaneous variance is equal to its mean. The correlation ρ between volatility and price shocks is zero. We also report sensitivities.

We use the same range of strikes and interest rates as before. With a strike of 110, immediate exercise is optimal. Whatever hedging strategy is used, it is liquidated immediately and there is zero hedge error.

⁹ To avoid excessively leveraged hedge portfolios, the holding of the European option under delta-gamma hedging is constrained to lie in the interval of $[-3, +3]$; this reduces the hedge error substantially on a small proportion of paths.

Table III: Hedge errors when hedging an American option using European options under both delta-gamma and robust hedges

		<i>Hedge Error - Standard deviation and 99%'ile</i>			
<i>Strike (K)</i>	<i>Delta-gamma hedge</i>	<i>Robust hedge with floor on imp vol of</i>			
		0%	5%	8%	10%
<i>r=2%</i>					
95	0.09	0.36	0.20	0.13	0.11
	(0.34)	(0.28)	(0.14)	(0.24)	(0.35)
100	0.18	0.54	0.33	0.20	0.19
	(0.47)	(0.59)	(0.38)	(0.40)	(0.43)
105	0.30	0.63	0.41	0.25	0.25
	(0.67)	(0.87)	(0.66)	(0.45)	(0.47)
<i>r=5%</i>					
95	0.23	0.66	0.41	0.21	0.18
	(0.80)	(0.40)	(0.23)	(0.30)	(0.62)
100	0.47	0.91	0.66	0.36	0.33
	(1.18)	(1.02)	(0.68)	(0.71)	(0.99)
105	0.77	0.90	0.75	0.42	0.45
	(1.18)	(1.30)	(1.14)	(0.68)	(0.65)
<i>r=10%</i>					
95	0.36	0.79	0.55	0.30	0.21
	(1.29)	(0.38)	(0.25)	(0.30)	(0.73)
100	0.78	1.04	0.94	0.50	0.40
	(1.79)	(1.25)	(0.91)	(0.77)	(1.19)
105	0.00	0.00	0.00	0.00	0.00
	(0.00)	(0.00)	(0.00)	(0.00)	(0.00)

The table shows the standard deviation of hedge error, and in parentheses, the 99% worst loss, from a strategy of writing a 1 year American option and hedging it to expiry or exercise. The underlying follows a Heston stochastic volatility process with parameters in the text. All options trade at fair value, and the American option is exercised optimally. The initial price of the underlying is 100. The runs consist of 100,000 simulations on a grid with 40 periods. Under delta-gamma hedging, the American option is hedged with a portfolio consisting of the underlying and the European option with the same strike and maturity as the American option being hedged; hedge ratios are computed using Black-Scholes to ensure that the hedged portfolio has zero delta and gamma.

The hedge error under delta-gamma hedging has a much lower standard deviation than the rational bounds hedge. The distribution of losses under the rational bounds hedge is

highly skewed; the maximum loss is in several of the cases less than one standard deviation below zero. The rational bounds hedge tends to have a smaller left hand tail than the delta-gamma hedge. For hedgers concerned with the possibility of large losses, the low and predictable maximum loss from the rational bounds hedge is attractive.

The performance of the rational bounds hedge can be much improved by imposing a floor on implied volatility. This greatly reduces the cost of the hedge (and hence its standard deviation) while not compromising its robustness, at least when the floor is reasonably low. If the floor is raised to 10% (the unconditional expected root mean square volatility) the standard deviation of the hedge error tends to diminish, but its robustness deteriorates.

Table IV shows the sensitivity of these numbers to the choice of parameters. The main results that come out are the sensitivity of the hedge error under delta-gamma hedging to the volatility of volatility. Halving σ roughly halves the hedge error; if the volatility of volatility were zero then, with the initial volatility being equal to its unconditional mean, volatility would be constant and the delta-gamma hedge would be perfect. By contrast, the rational bounds hedge is virtually unaffected by the volatility of volatility. It has limited impact on either the cost of the rational bounds hedge or the cost of the American option, and hence leaves the maximum loss largely unaffected. The robust hedges with positive floors are more heavily affected by the volatility of volatility since the higher it is, the greater the likelihood that the liquidation value of the hedge portfolio is insufficient to meet the exercise value of the American option.

The rate of mean reversion has little impact on the robust hedges, but it does have a noticeable impact on the delta-gamma hedge. In the table the lower value of κ is accompanied by a reduction in volatility of volatility to keep the unconditional variance of variance constant. As the rate of mean reversion tends to infinity, the process tends towards a jump diffusion process, the conditional expectation of future volatility is constant, and the delta gamma hedge works well.

With volatility (initial and long run mean) doubled, and σ increased to maintain the coefficient of variation of the variance of returns, the delta-gamma hedge error is essentially unaltered, while the error on the robust hedge is increased substantially.

Nevertheless it remains the case that by raising the floor, the robust hedge has a substantially smaller left tail than the delta-gamma hedge.

Table IV: Sensitivity of hedge error to parameter choice

<i>Strike</i> (<i>K</i>)	<i>Hedge Error - Standard deviation and 99%'ile</i>				
	<i>Delta-gamma</i> <i>hedge</i>	<i>Robust hedge with floor on imp vol of</i>			
		0%	5%	8%	10%
<i>Base Case</i>	0.47 (1.18)	0.91 (1.02)	0.66 (0.68)	0.36 (0.71)	0.33 (0.99)
<i>Low vol of vol</i> $\sigma = 0.1(0.2)$	0.26 (0.67)	0.98 (1.09)	0.70 (0.77)	0.30 (0.30)	0.19 (0.59)
<i>Low mean reversion</i> $\kappa, \sigma = 0.5, 0.1(2, 0.2)$	0.52 (1.57)	0.93 (1.08)	0.70 (0.79)	0.40 (0.77)	0.35 (1.14)
<i>High volatility</i> $\sqrt{\theta}, \sqrt{v_0} = 20\%(10\%); \sigma = 0.1(0.2)$	0.48 (1.17)	1.27 (1.39)	1.28 (1.32)	0.91 (1.09)	0.78 (0.88)
<i>High initial volatility</i> $v_0 = 14\%(10\%)$	0.49 (1.46)	0.96 (1.23)	0.73 (0.98)	0.44 (0.74)	0.32 (0.87)
<i>Negative correlation</i> $\rho = -0.5(0)$	0.38 (1.09)	0.91 (0.94)	0.66 (0.66)	0.35 (0.56)	0.25 (0.76)
<i>Coarser grid</i> 20 periods (40)	0.45 (1.08)	0.92 (1.04)	0.68 (0.76)	0.37 (0.68)	0.35 (0.84)

The table shows the standard deviation of hedge error, and in parentheses, the 99% worst loss, from a strategy of writing a 1 year American option with strike of 100 and hedging it to expiry or exercise. The initial price of the underlying is 100. The runs consist of 100,000 simulations. The interest rate is 5%. The underlying follows a Heston stochastic volatility process with parameters in the text. All options trade at fair value, and the American option is exercised optimally. Under delta-gamma hedging, the American option is hedged with a portfolio consisting of the underlying and the European option with the same strike and maturity as the American option being hedged; hedge ratios are computed using Black-Scholes to ensure that the hedged portfolio has zero delta and gamma.

The level of initial volatility, of correlation between price and volatility shocks, and the coarseness of the grid do not appear to have a substantial effect on the conclusions.

6. CONCLUSIONS

In this paper we have taken seriously the American option holder's right to adjust the exercise strategy in the light of information about future asset returns received after the option has been acquired. Standard models used for pricing American options tend to make highly restrictive assumptions about future information. In the extreme case of Black-Scholes, the holder receives no new information at all about the distribution of future returns.

We have shown how to construct the cheapest strategy that dominates the American option. It involves buying a portfolio of European options, and delta hedging to extract their intrinsic value once the American option is exercised. The cost of the European portfolio is the tightest model-free bound on the price of the American option. The bound is attainable in the sense that there are processes that are consistent with the initial set of European option prices under which the American option has a value equal to that bound. In the case of the American put, the bound on the American premium can be three or four times the Black-Scholes value.

The type of process that leads to the attaining of the bound was identified. It has two states, one of high and one of low volatility. The American option is exercised when the asset moves from the high volatility state to the low volatility state. The Single Jump Process is an extreme example of this type, and has out-of-the-money puts trading on their upper bounds.

If plausible restrictions can be put on the level of implied volatility in future, the bounds on the American option can be tightened. We showed how this could be done. With a floor on future implied volatility set at half the current forward volatility, the bound on the American premium is reduced by around 20% on average.

The search for option pricing bounds has a useful by-product: the identification of hedging strategies that are not dependent on highly specified models, and which are therefore robust. We use simulation to show that these robust strategies perform well

compared with more standard delta-gamma hedging strategies, particularly from the perspective of agents who put heavy weight on avoiding large losses.

REFERENCES

Andersen, L., and J. Andreasen, 2001, “Factor Dependence of Bermudan Swaptions: fact or fiction?”, *Journal of Financial Economics*, 62(1), 3–37.

Andersen, L., and M. Broadie, 2004, “Primal-Dual Simulation Algorithm for Pricing Multidimensional American Options”, *Management Science*, 50(9), 1222–1234.

Avellaneda, M., A. Levy, and A. Paras, 1995, “Pricing and hedging derivative securities in markets with uncertain volatilities”, *Applied Mathematical Finance*, 2, 73–88.

Bernardo, A.E., and O. Ledoit, 2000, “Gain, Loss, and Asset Pricing”, *Journal of Political Economy*, 108(1), 144–172.

Britten-Jones, M., and A. Neuberger, 2000, “Option Prices, Implied Price Processes and Stochastic Volatility”, *Journal of Finance*, 55(2), 839-866.

Broadie, M., and J. B. Detemple, 2004, “Option Pricing: Valuation Models and Applications”, *Management Science*, 50(9), 1145–1177.

Brown, H., Hobson, D. and L.C.G. Rogers, 2001, “Robust Hedging of Barrier Options”, *Mathematical Finance*, 11, 285–314.

Buraschi, A., and F. Corielli, 2005, “Risk management implications of time-inconsistency: Model updating and recalibration of no-arbitrage models”, *Journal of Banking and Finance*, 29(11), 2883-2907.

Cochrane, J. H., and J. Saá-Requejo, 2000, “Beyond Arbitrage: Good-Deal Asset Price Bounds in Incomplete Markets”, *Journal of Political Economy*, 108(1), 79-119.

Davis, M. H. A., and D. G. Hobson, 2007, “The Range of Traded Option Prices”, *Mathematical Finance*, 17(1), 1-14.

Derman, E., and I. Kani, 1994, “Riding on a Smile”, *Risk*, 7, 32-39.

Derman, E., D. Ergener, and I. Kani, 1995, "Static Options Replication," *Journal of Derivatives*, 2, 78-95.

Dupire, B., 1994, "Pricing with a smile", *Risk*, 7, 18-20.

Green T, C., and S. Figlewski, 1999, "Market Risk and Model Risk for a Financial Institution Writing Options", *Journal of Finance*, 54, 1465-1499.

Haugh, M., and L. Kogan, 2004, "Pricing American Options: A Duality Approach", *Operations Research*, 52(2), 258-270.

Heston, S., 1993, "A closed-form solutions for options with stochastic volatility", *Review of Financial Studies*, 6, 327-343. .

Longstaff, F., P. Santa-Clara and E. Schwartz, 2001, "Throwing away a billion dollars: the cost of suboptimal exercise strategies in the swaption market", *Journal of Financial Economics*, 62(1), 39-66.

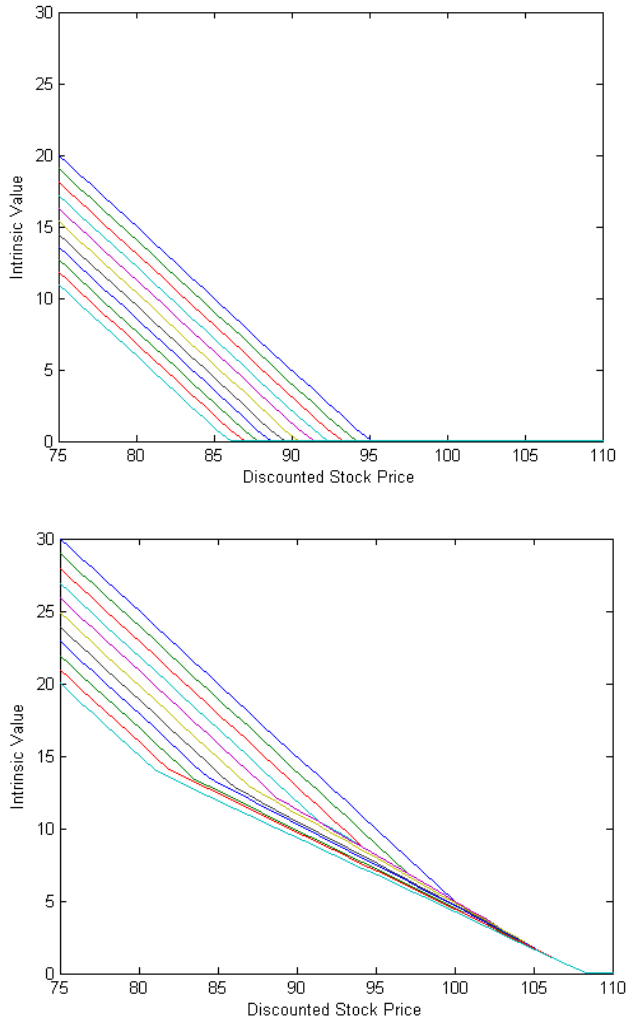
Merton, R. C., 1973, "The Theory of Rational Option Pricing", *Bell Journal of Economics*, 4, 141-183.

Rogers, L. C. G., 2002, "Monte Carlo Valuation of American options", *Mathematical Finance*, 12 (3), 271-286.

Rumsfeld, D., 2002, Defense Secretary Rumsfeld Press Conference at NATO Headquarters, Brussels, Belgium, 6 June 2002, <http://www.defenselink.mil/transcripts/transcript.aspx?transcriptid=3490>

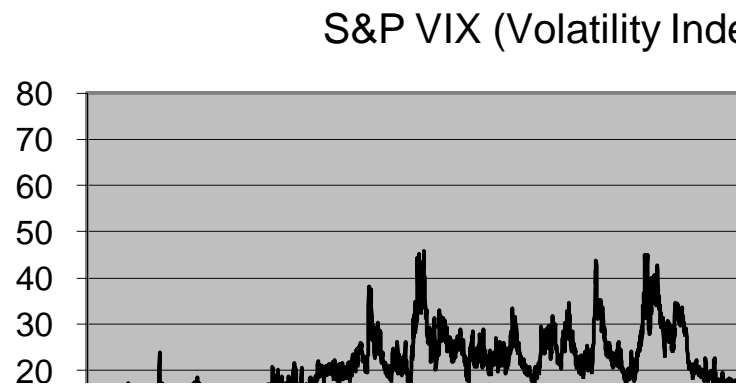
Svenstrup, M., 2005, "On the suboptimality of single-factor exercise strategies for Bermudan swaptions", *Journal of Financial Economics*, 78(3), 651-684.

Figure 1: The Cheapest Dominating Portfolios for an American Put



The two charts show the European options that dominate an American put option. In the top chart the strike price of the American option is 95, while in the bottom it is 105. The difference between successive lines shows the cash flow that period as a function of the asset price. The other parameters are: initial asset price 100, risk free rate 10%, option maturity 1 year, European options have an implied annual volatility (using a binomial model with a mesh of 1%) of 10%.

Figure 2: The dynamics of implied volatility



The VIX index is the implied volatility of a synthetic at-the-money 30 day option on the S&P500 index, and it is traded on the Chicago Board Options Exchange. It is measured in annualized percentage points. Data collected from Yahoo Finance.

APPENDIX

Proof of Proposition 2:

Rewrite the problem LP2 as:

Find $\{\mathbf{F}, \mathbf{G}, \mathbf{H}\} \geq \mathbf{0}$ that maximizes $\sum_{j,t} a_{j,t} f_{j,t}$ subject to:

$$\begin{aligned}
 1) \quad & \sum_m (g_{j,m,t} + h_{j,m,t}) \geq p_{j,t}; & 2) \quad & -\sum_m (g_{j,m,t} + h_{j,m,t}) \geq -p_{j,t}; \\
 3) \quad & \sum_m (g_{m,j,t} + h_{m,j,t}) \geq p_{j,t+1} \quad (t < T); & 4) \quad & -\sum_m (g_{m,j,t} + h_{m,j,t}) \geq -p_{j,t+1} \quad (t < T); \\
 5) \quad & \sum_m (x_m - x_j) g_{j,m,t} \geq 0; & 6) \quad & -\sum_m (x_m - x_j) g_{j,m,t} \geq 0; \\
 7) \quad & \sum_m (x_m - x_j) h_{j,m,t} \geq 0; & 8) \quad & -\sum_m (x_m - x_j) h_{j,m,t} \geq 0; \\
 9) \quad & f_{j,t} - \sum_m (h_{j,m,t} - h_{m,j,t-1}) \geq 0; & 10) \quad & -f_{j,t} + \sum_m (h_{j,m,t} - h_{m,j,t-1}) \geq 0;
 \end{aligned}$$

for $j = 1, \dots, J$ and $t = 1, \dots, T$, with $h_{m,j,t} \equiv 0$ when $t = 0$.

Compare this with the linear program in standard form:

LP3: Find $\mathbf{x} \geq \mathbf{0}$ that maximizes $z = \mathbf{c}^T \mathbf{x}$ subject to $\mathbf{A} \mathbf{x} \leq \mathbf{b}$,
 where \mathbf{x} , \mathbf{c} are $n \times 1$ vectors, \mathbf{b} is $m \times 1$, and \mathbf{A} is $m \times n$.

It can be seen that n , the number of variables, is $JT + 2J^2T$, while m , the number of constraints, is $10JT - 2J$. The vector of variables \mathbf{x} consists of three blocks: the elements of \mathbf{F} , then the elements of \mathbf{G} , and finally the elements of \mathbf{H} . The vector \mathbf{c} consists of the elements of \mathbf{A} , followed by $2J^2T$ zeros. The vector \mathbf{b} consists of the elements of \mathbf{P} , followed by $-\mathbf{P}$, then \mathbf{P} again (minus its first row) and the negative of this, followed by $6JT$ zeros. \mathbf{A} is not readily describable in detail, but its general structure is illustrated below where the blocks of zeros are identified by a minus sign, and the blocks with some non-zero elements are denoted by a plus sign:

JT	+	-	+
JT	+	-	+
JT	-	-	+
JT	-	-	+
JT	-	+	-
JT	-	+	-
$JT-J$	-	+	+
$JT-J$	-	+	+
JT	-	+	+
JT	-	+	+
	JT	J^2T	J^2T

The dual of LP3 is:

LP4: Find $\mathbf{y} \geq \mathbf{0}$ that minimizes $w = \mathbf{b}^T \cdot \mathbf{y}$ subject to $\mathbf{A}^T \cdot \mathbf{y} \geq \mathbf{c}$.

If we break \mathbf{y} into 10 blocks so $\mathbf{y} = (\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^{10})$ then LP4 can be written as:

LP5:

Find $\{\mathbf{y}^1, \dots, \mathbf{y}^{10}\} \geq \mathbf{0}$ that minimizes $\sum_{j,t} (y_{j,t}^1 - y_{j,t}^2 + y_{j,t}^3 - y_{j,t}^4) p_{j,t}$ subject to:

- 1) $(y_{j,t}^9 - y_{j,t}^{10}) \geq a_{j,t};$
 - 2) $(y_{j,t}^1 - y_{j,t}^2 + y_{j,t}^3 - y_{j,t}^4) + (x_m - x_j)(y_{j,t}^5 - y_{j,t}^6) \geq 0;$
 - 3) $(y_{j,t}^1 - y_{j,t}^2 + y_{j,t}^3 - y_{j,t}^4) + (x_m - x_j)(y_{j,t}^7 - y_{j,t}^8) + (y_{j,t}^{10} - y_{j,t}^9) - (y_{m,t+1}^{10} - y_{m,t+1}^9) \geq 0;$
- where $y_{j,t}^3 = y_{j,t}^4 = 0$ when $t = 1$ and $y_{m,t}^9 = y_{m,t}^{10} = 0$ when $t = T + 1$,
for $j, m = 1, \dots, J$ and $t = 1, \dots, T$.

The bracketing shows where terms are always associated. Define the four $J \times T$ matrices

C, **D**, **V** and **E** as:

$$\begin{aligned}
\mathbf{C} &\equiv \mathbf{y}^5 - \mathbf{y}^6; \\
\mathbf{D} &\equiv \mathbf{y}^7 - \mathbf{y}^8; \\
\mathbf{V} &\equiv \mathbf{y}^9 - \mathbf{y}^{10}; \\
\mathbf{E} &\equiv \mathbf{y}^1 - \mathbf{y}^2 + \mathbf{y}^3 - \mathbf{y}^4.
\end{aligned}$$

Then LP5 becomes:

LP6:

Find $\{\mathbf{C}, \mathbf{D}, \mathbf{V}, \mathbf{E}\}$ that minimizes $\sum_{j,t} e_{j,t} p_{j,t}$ subject to:

- 1) $v_{j,t} \geq a_{j,t}$;
- 2) $e_{j,t} + (x_m - x_j) c_{j,t} \geq 0$;
- 3) $e_{j,t} + (x_m - x_j) d_{j,t} - v_{j,t} + v_{m,t+1} \geq 0$;

where $v_{m,t} \equiv 0$ when $t = T + 1$,

for $j, m = 1, \dots, J$ and $t = 1, \dots, T$.

Consider constraint (2). By setting $m = j$ it is seen to imply that $\mathbf{E} \geq \mathbf{0}$. But that is all that (2) implies, because this is the only place in the LP6 where \mathbf{C} is mentioned, so provided $\mathbf{E} \geq \mathbf{0}$, we can ensure that (2) is satisfied by setting $\mathbf{C} = \mathbf{0}$. The solution to LP6 remains unchanged if constraint (2) is replaced by $e_{j,t} \geq 0$, and \mathbf{C} is dropped from the list of variables. But then the problem is identical to LP1.

Proof of Proposition 4

1. Suppose that for some j and t , $\lambda_{j,t} > \lambda_{j,t}^F$. The following is an arbitrage:

- at time 0, buy $\alpha \equiv \frac{x_{j+1} - x_{j-1}}{(x_{j+1} - x_j)(x_j - x_{j-1})}$ calendar spreads, where each spread is long the call $c_{j,t+1}$ and short the call $c_{j,t}$. Also write $\lambda_{j,t}$ Arrow-Debreu securities $p_{j,t}$. By assumption, this provides strictly positive cash flow.

- at time t , pay any liabilities that have fallen due and lay on the following hedges:
 - if $S_t > x_j$ short a units of the underlying;
 - if $S_t = x_j$ short $1/(x_j - x_{j-1})$ of the underlying, and sell one local volatility contract;
 - if $S_t < x_j$ do nothing;
 - at time $t+1$, liquidate the portfolio. It can readily be shown that the portfolio is either a wash (if the asset price has moved at most one node over the preceding period) or strictly positive if it has moved by more.
2. The restrictions are consistent with the martingale process Q in the sense that the price of any claim is equal to its expected pay-off under Q . Hence there cannot be an arbitrage.
3. Define v, v^c, d, e , and y recursively as follows:

$$\begin{aligned}
 v_{j,T} &= \text{Max}\{a_{j,T}, 0\}; v_{j,T}^c = 0; \\
 v_{j,t-1}^c &= v_{j,t} + \lambda_{j,t-1} \left(\frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}} v_{j+1,t} - v_{j,t} + \frac{x_{j+1} - x_j}{x_{j+1} - x_{j-1}} v_{j-1,t} \right); \\
 (1) \quad v_{j,t} &= \text{Max}\{v_{j,t}^c, a_{j,t}\}; \\
 d_{j,t} &= -(v_{j+1,t+1} - v_{j-1,t+1}) / (x_{j+1} - x_{j-1}); \\
 e_{j,t} &= v_{j,t} - v_{j,t}^c; \\
 y_{j,t} &= \frac{1}{u+1} v_{j+1,t+1} - v_{j,t+1} + \frac{u}{u+1} v_{j-1,t+1}.
 \end{aligned}$$

v is the conventional American option valuation, v^c is the continuation value, and e is the difference between the two. To show that (1) is a feasible solution to LP*, note that constraints 1 and 2 of LP* are satisfied trivially. For $y \geq 0$ (constraint 4), v must be convex in x . Constraint 3 is satisfied exactly at $m = j-1, j$ and $j+1$. For it to be satisfied for other values of m it is again sufficient that v be convex. The

convexity of v can be proved by induction. Since a is convex, and the maximum of two convex functions is convex, $v_{\cdot,T}$ is convex. λ lies between 0 and 1. If $v_{\cdot,t}$ is convex, then:

$$\begin{aligned}
 (2) \quad & v_{j,t-1}^c \leq \frac{x_j - x_{j-1}}{x_{j+1} - x_{j-1}} v_{j+1,t} + \frac{x_{j+1} - x_j}{x_{j+1} - x_{j-1}} v_{j-1,t} \quad (\text{max when } \lambda_{j,t-1} = 1); \\
 & v_{j+1,t-1}^c \geq v_{j+1,t} \quad (\text{min when } \lambda_{j+1,t-1} = 0); \\
 & v_{j-1,t-1}^c \geq v_{j-1,t} \quad (\text{min when } \lambda_{j-1,t-1} = 0).
 \end{aligned}$$

so $v_{\cdot,t-1}^c$ is convex. $v_{\cdot,t-1}$ is the maximum of $v_{\cdot,t-1}^c$, which we have just shown to be convex, and $a_{\cdot,t-1}$, which is convex by assumption. So $v_{\cdot,t-1}$ too must be convex.

Proposition 1 applies and $v_{0,0}$ is an upper bound on the price of the American option. But it is also the expected value of the option under the trinomial process Q so it must be the least upper bound.